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ON THE EXISTENCE OF THE VALUE OF A DIFFERENTIAL GAME OF SPECIFIED DURATION

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A natural extension of Isaac's formulation [1] leads to differential games which are described by differential equations with aftereffect and as regards information are determined by information on the whole prehistory of the game. In this paper relating closely to the investigations in [2 - 10], the emphasis is on the proof of the existence theorem for the value of the differential game in such an extended formulation. We examine a differential game of prescribed duration with a payoff given in the form of a functional on the game's trajectory. The differential game is approximated by a certain family of multistage games. We consider two sequences (minimax and maximin) comprised of the minimax and the maximin values of the payoff in the multistage games. Conditions are obtained under whose fulfillment these sequences converge and their limits are equal. The proof of the convergence of the minimax and maximin sequences is carried out by a scheme suggested by Fleming [2]. The proof of the equality of the limits of these sequences is based on the results of Krasovskii and Subbotin [8, 9].

1. Let X be a Euclidean space, P and Q compact subsets of X, $[0, \vartheta]$ a specified time interval, $C[0, \tau]$ the collection of functions $x = x(s)$ continuous on $[0, \tau]$

$$\|x\| = \sqrt{xx}, \qquad \|x(s)\|_\tau = \max_{0 \leqslant s \leqslant \tau} \|x(s)\|$$

$$\mathrm{P} = \{(t, x(s: 0 \leqslant s \leqslant t)) : t \in [0, \vartheta], x(s) \in C[0, t]\}$$

$$\mathrm{P}[t] = \{(t, x(s: 0 \leqslant s \leqslant t)) : x(s) \in C[0, t]\}$$

Let f be a function, continuous on $\mathrm{P} \times P \times Q$, with a range of the values in X, F be a functional continuous on $C[0, \vartheta]$. The sets P, Q are the players' control regions. ϑ is the prescribed instant of the game completion and f is the right-hand side of the equation of motion

$$x^{\cdot}(\ ) = f[t, x(s: 0 \leqslant s \leqslant t), u, v] \qquad (u \in P,\ v \in Q) \qquad (1.1)$$

Player I, having the control $u \in P$, strives to minimize the value of the payoff functional F, player II, with control $v \in Q$ at his disposal, pursues a contrary purpose. The players know the function f, the sets P, Q, the functional F, and the instant ϑ of the completion of the game. Both players are given full information during the game: at each current instant t they are informed of the value of the variable t and of the portion $x(s: 0 \leqslant s \leqslant t)$ of the trajectory realized by this instant.

The following requirements are imposed on the function f, the sets P, Q, and the functional F:

1. For a fixed $p \in P$, for any $e \in X$,

$$\min_{u\in P}\max_{v\in Q} ef[p, u, v] = \max_{v\in Q}\min_{u\in P} ef[p, u, v]$$

2. For every $p \in P$ the sets $f[p, P, v]$, $f[p, u, Q]$ are convex.
3. There exists $K > 0$ such that

$$\| f[t, y(s: 0 \leq s \leq t), u, v] - f[t, z(s: 0 \leq s \leq t), u, v] \| \leq \\ \leq K \| y(s) - z(s) \|_t$$

for all $t \in [0, \vartheta]$, $y(s), z(s) \in C[0, \vartheta]$, $u \in P$, $v \in Q$.

4. The function f is continuous on $P \times P \times Q$ in the sense that

$$f[t_n, x_n(s: 0 \leq s \leq t_n), u_n, v_n] \to f[t, x(s: 0 \leq s \leq t), u, v]$$

as

$$t_n \to t, \ \|x_n(s) - x(s)\|_\vartheta \to 0, \ u_n \to u, \ v_n \to v.$$

5. There exists $M > 0$ such that $\| f \| \leq M$ on $P \times P \times Q$.
6. There exists $L > 0$ such that

$$|F[y(s: 0 \leq s \leq \vartheta)] - F[z(s: 0 \leq s \leq \vartheta)]| \leq L \| y(s) - z(s) \|_\vartheta$$

for all $y(s), z(s) \in C[0, \vartheta]$.

Such is the informal description of the differential game Γ being analyzed. However, a direct investigation of the existence problem in the continuous differential game meets with insurmountable difficulties connected with the necessity of restricting the players to such behaviors $u = u[t, x(s: 0 \leq s \leq t)]$, $v = v[t, x(s: 0 \leq s \leq t)]$ which would guarantee the integrability of Eq. (1.1). These difficulties can be overcome by going to a discrete formulation. Here the original differential game is approximated by a certain family of multistage games. Two sequences (minimax and maximin) comprised of the minimax and the maximin payoff values of the multistage games are considered. If these sequences converge to some common limit, this limit value is called the generalized value of the original differential game Γ

2. Let Σ be the collection of coverings of the interval $[0, \vartheta]$ by a finite system of contiguous intervals $[t_{i-1}, t_i]$

$$0 = t_0 < t_1 < \ldots < t_n = \vartheta$$

Let σ be the general element of set Σ, $l(\sigma)$ be the number of intervals $[t_{i-1}, t_i]$ in covering σ, $|\sigma|$ be the largest of the lengths $\Delta_i = t_i - t_{i-1}$, $P_i = P[t_i]$. Let A, B be arbitrary sets. The collection of single-valued mappings of set A into set B is denoted $[A \to B]$.

Let us formulate the definition of the family $\{\Gamma_\sigma : \sigma \in \Sigma\}$ of multistage games Γ_σ by which the original differential game Γ is approximated. The multistage game Γ_σ of duration $n = l(\sigma)$ stages is described by the equation

$$\begin{gathered} x(t) = x(t_{i-1}) + (t - t_{i-1}) f[t_{i-1}, x(s: 0 \leq s \leq t_{i-1}), u_i, v_i] \\ (t \in [t_{i-1}, t_i], \ u_i \in P, \ v_i \in Q, \ 1 \leq i \leq n) \end{gathered} \tag{2.1}$$

The payoff is given by the functional F. Player I, having the control $u \in P$, strives to minimize the value of functional F; player II, having control $v \in Q$ at his disposal; pursues the contrary purpose. The players know the function f, the sets P, Q, the covering σ and the functional F. Both players are supplied with full information during the game: at each stage i they are informed about the position $(t_{i-1}, x(s: 0 \leqslant s \leqslant t_{i-1}))$ realized by the instant of this stage. This information permits the players to shape their own behavior in the form of the functions

$$u_i = u_i [t_{i-1}, x(s: 0 \leqslant s \leqslant t_{i-1})] \in [\mathrm{P}_{i-1} \to P]$$
$$v_i = v_i [t_{i-1}, x(s: 0 \leqslant s \leqslant t_{i-1})] \in [\mathrm{P}_{i-1} \to Q]$$

The sequence $\{u_1, \ldots, u_n\}$ $(\{v_1, \ldots, v_n\})$ of such functions is called the strategy of player I (II).

Together with Γ_σ we consider the multistage games $\Gamma_\sigma^{\pm}$. The majorant Γ_σ^+ (minorant Γ_σ^-) is defined analogously to Γ_σ with the only difference that here at each stage i the second (first) player chooses his own current control having already been informed of the choice made by his opponent. The strategy of I (II) in Γ_σ^+ is the sequence of mappings

$$u_i \in [\mathrm{P}_{i-1} \to P] \quad (v_i \in [\mathrm{P}_{i-1} \times P \to Q]) \quad (1 \leqslant i \leqslant n)$$

The strategy of I (II) in Γ_σ^- is the sequence

$$u_i \in [\mathrm{P}_{i-1} \times Q \to P] \quad (v_i \in [\mathrm{P}_{i-1} \to Q]) \quad (1 \leqslant i \leqslant n)$$

In the continuous formulation, to the game Γ_σ^+ (Γ_σ^-) there corresponds the differential game Γ^+ (Γ^-) with the first (second) player having discrimination [11].

We see that the minimax value of functional F in game Γ_σ

$$\min_{u_1 \in [\mathrm{P}_0 \to P]} \ldots \min_{u_n \in [\mathrm{P}_{n-1} \to P]} \max_{v_1 \in [\mathrm{P}_0 \to Q]} \ldots \max_{v_n \in [\mathrm{P}_{n-1} \to Q]} F[x(s: 0 \leqslant s \leqslant \vartheta)]$$

coincides with the minimax value of F in game Γ_σ^+

$$\min_{u_1 \in [\mathrm{P}_0 \to P]} \ldots \min_{u_n \in [\mathrm{P}_{n-1} \to P]} \max_{v_1 \in [\mathrm{P}_0 (\times) P \to Q]} \ldots \max_{v_n \in [\mathrm{P}_{n-1} (\times) P \to Q]} F[x(s: 0 \leqslant s \leqslant \vartheta)]$$

We denote this common minimax value by $V_\sigma^+ = V_\sigma^+(x_0)$. The maximin payoff value common for games Γ_σ, Γ_σ^- is denoted $V_\sigma^- = V_\sigma^-(x_0)$.

Let us agree to denote arbitrary functions from $C[0, \tau]$ by $y(t)$ or $z(t)$, keeping the notation $x(t)$ for the trajectories of Eq. (2.1). We set

$$\mathrm{VAL}_k^+ = \min_{u_1 \in P} \max_{v_1 \in Q} \ldots \min_{u_k \in P} \max_{v_k \in Q}, \quad \mathrm{VAL}_k^- = \max_{v_1 \in Q} \min_{u_1 \in P} \ldots \max_{v_k \in Q} \min_{u_k \in P}$$

$$\mathrm{VAL}^{\pm} = \mathrm{VAL}_1^{\pm}$$

T h e o r e m 1. Let Conditions 4 – 6 be fulfilled. Then in the games $\Gamma_\sigma^{\pm}$ there exists a saddle point, and the values $V_\sigma^{\mp}$ of these games satisfy the relation

$$V_\sigma^{\pm}(x_0) = \mathrm{VAL}_n^{\pm} F[x(s: 0 \leqslant s \leqslant \vartheta)] \qquad (n = l(\sigma)) \tag{2.2}$$

P r o o f. The reasoning is based on the consideration of the funtions $V_{\sigma,i}^{\pm}$ $(0 \leqslant i \leqslant n)$ defined by the recurrence relations

$$V_{\sigma,n}^{\pm}[y(s: 0 \leqslant s \leqslant \vartheta)] = F[y(s: 0 \leqslant s \leqslant \vartheta)]$$

$$V_{\sigma,i-1}^{\pm}[y(s:0\leqslant s\leqslant t_{i-1})]=\mathrm{VAL}^{\pm}V_{\sigma,i}^{\pm}[y_*(s:0\leqslant s\leqslant t_i)]$$

where $y_*(t)=y_*(t;u,v)$ is a continuous prolongation of the function $y(t)\in C[0,t_{i-1}]$ on the interval $[0,t_i]$, given on $[t_{i-1},t_i]$ by the formula

$$y_*(t)=y(t_{i-1})+(t-t_{i-1})f[t_{i-1},y(s:0\leqslant s\leqslant t_{i-1}),u,v]$$

By the very same inductive scheme by which the proof is carried out of the Zermelo-Neumann existence theorem in positional games [12], we convince ourselves that $V_{\sigma,i}^{\pm}[y(s:0\leqslant s\leqslant t_i)]$ is the value (payoff) of the game Γ_σ^{+} corresponding to the position $(t_i, y(s:0\leqslant s\leqslant t_i))$ as the initial position. The payoff $V_{\sigma,0}^{\pm}$ is simultaneously the minimax and the maximin value of the payoff in $\Gamma_{\sigma,0}^{\pm}$. Therefore, $V_{\sigma,0}^{\pm}[x_0]=V_\sigma^{\pm}(x_0)$. Now (2.2) ensues from the definition of the functions $V_{\sigma,i}^{\pm}$.

3. Here, after a number of auxiliary assertions, we establish that when Conditions 2 – 6 are fulfilled there exist the generalized values

$$V^{\pm}(x)=\lim V_\sigma^{\pm}(x)\quad\text{as}\quad|\sigma|\to 0 \tag{3.1}$$

of the differential games $\Gamma^{\pm}$.

Lemma 1. Let φ_1, φ_2 be scalar functions, continuous on $(P\times Q)^k$. Let a be the maximum deviation $|\varphi_1-\varphi_2|$ on $(P\times Q)^k$. Then

$$|\mathrm{VAL}_k^{\pm}\varphi_1(u_1,v_1,\ldots,u_k,v_k)-\mathrm{VAL}_k^{\pm}\varphi_2(\cdot)|\leqslant a$$

For $k=1$ the proof of the lemma is immediate [13], while in the general case it is achieved by induction. We say that the covering $\sigma'\in\Sigma$ contains the covering $\sigma\in\Sigma$, if every interval $[t_{i-1},t_i]\in\sigma$ can be represented as the sum of $m_i\geqslant 1$ of intervals $[t_{i,j-1},t_{i,j}]\in\sigma'$ in such a way that

$$t_{i-1}=t_{i,0}<t_{i,1}<\ldots<t_{i,m_i}=t_{i+1}\quad(1\leqslant i\leqslant n=l(\sigma))$$

$$\Delta_{i,1}+\ldots+\Delta_{i,m_i}=\Delta_i\quad(\Delta_{i,j}=t_{i,j}-t_{i,j-1})$$

$$m_1+\ldots+m_n=m=l(\sigma')$$

Let $\sigma'\supseteq\sigma$. Consider the function

$$V_{\sigma,\sigma',n}^{\pm}[y(s:0\leqslant s\leqslant\vartheta)]=F[y(s:0\leqslant s\leqslant\vartheta)]$$

$$V_{\sigma,\sigma',i-1}^{\pm}[y(s:0\leqslant s\leqslant t_{i-1})]=\mathrm{VAL}_{m_i}^{\pm}V_{\sigma,\sigma',i}^{\pm}[y^*(s:0\leqslant s\leqslant t_i)]$$

Here and subsequently $y^*(t)=y^*(t;u_1,v_1,\ldots,u_{m_i},v_{m_i})$ is a continuous prolongation of the function $y(t)\in C[0,t_{i-1}]$ on the interval $[0,t_i]$ defined on $[t_{i-1},t_i]$ by the formula

$$y^*(t)=y(t_{i-1})+(t-t_{i-1})\sum_{1\leqslant j\leqslant m_i}(\Delta_{i,j}/\Delta_i)f[t_{i-1},y(s:0\leqslant s\leqslant t_{i-1}),u_j,v_j]$$

We set $V_{\sigma,\sigma'}^{\pm}=V_{\sigma,\sigma',0}^{\pm}$. We note that if $\sigma'=\sigma$, then

$$V_{\sigma,\sigma',i}^{\pm}=V_{\sigma,i}^{\pm},\qquad V_{\sigma,\sigma'}^{\pm}=V_\sigma^{\pm}$$

Lemma 2. Let Conditions 3 – 6 be fulfilled. Let $\sigma'\supseteq\sigma$. Then the functions $V_{\sigma,\sigma',i}^{\pm}$ satisfy a Lipschitz condition on $C[0,t_i]$ with the constant $Le^{K\vartheta}$.

Proof. Let us show that for each $0\leqslant i\leqslant n$

$$| V^{\pm}_{\sigma, \sigma', i} [y (s: 0 \leqslant s \leqslant t_i)] - V^{\pm}_{\sigma, \sigma', i} [z (s: 0 \leqslant s \leqslant t_i)] \leqslant a_i \| y (s) - z (s) \|_{t_i} \quad (3.2)$$

where

$$a_n = L, \qquad a_{i-1} = (1 + K\Delta_i) a_i, \qquad 1 \leqslant i \leqslant n \quad (3.3)$$

This is true for $i = n$. Reasoning inductively, we assume that (3.2) is valid for some i. Under this assumption we prove (3.2) for $i - 1$. By Condition 3, for every $t \in [t_{i-1}, t_i]$

$$\| y^* (t) - z^* (t) \| \leqslant \| y(t_{i-1}) - z (t_{i-1}) \| + \sum_{1 \leqslant j \leqslant m_i} \Delta_{i,j} K \| y (s) - z (s) \|_{t_{i-1}} =$$

$$= \| y (t_{i-1}) - z (t_{i-1}) \| + \Delta_i K \| y (s) - z (s) \|_{t_{i-1}} \leqslant (1 + K\Delta_i) \| y (s) - z (s) \|_{t_{i-1}}$$

From this and from the inductive assumption we find that

$$| V^{\pm}_{\sigma, \sigma', i} [y^* (s: 0 \leqslant s \leqslant t_i)] - V^{\pm}_{\sigma, \sigma', i} [z^* (s: 0 \leqslant s \leqslant t_i)] | \leqslant$$

$$\leqslant a_i \max \{ \| y (s) - z (s) \|_{t_{i-1}}, \quad (1 + K\Delta_i) \| y (s) - z (s) \|_{t_{i-1}} \} =$$

$$= a_i (1 + K\Delta_i) \| y (s) - z (s) \|_{t_{i-1}}$$

for any $(u_1, v_1, \ldots, u_{m_i}, v_{m_i}) \in (P \times Q)^{m_i}$. The validity of (3.2) for $i - 1$ is explained by Lemma 1. Relation (3.2) is established. All $\Delta_i > 0$, $\Delta_1 + \ldots + \Delta_n = \vartheta$, therefore, as a consequence of (3.3),

$$a_n \leqslant a_{n-1} \leqslant \ldots \leqslant a_0 = (1 + K\Delta_1) \ldots (1 + K\Delta_n) L \leqslant L (1 + K\vartheta / n)^n \leqslant Le^{K\vartheta}$$

Replacing a_i in (3.2) by the constant $Le^{K\vartheta}$, we arrive at the lemma's assertion.

Let $A (x)$ be the collection of functions $z (t)$ absolutely continuous on $[0, \vartheta]$, constrained by $z (0) = x$, $\| z (t) \| \leqslant M$. By $\omega (\lambda, x)$ we denote the maximum of the deviation

$$\| f [t, z (s : 0 \leqslant s \leqslant t), u, v] - f [\tau, z (s : 0 \leqslant s \leqslant \tau), u, v] \|$$

over all $| t - \tau | \leqslant \lambda$, $z (s) \in A (x)$, $(u, v) \in P \times Q$. In accordance with Condition 4 the function $\omega (\lambda, x)$ is continuous on $[0, \vartheta] \times X$. Furthermore, $\omega (0, x) \equiv 0$.

Lemma 3. Let conditions 3 – 6 be fulfilled. Let $\sigma' \supseteq \sigma$. Then for any $x \in X$.

$$| V_{\sigma'}^{\pm} (x) - V^{\pm}_{\sigma, \sigma'} (x) | \leqslant L\Omega$$

$$\Omega = 2Me^{K\vartheta} | \sigma | + [(e^{K\vartheta} - 1) / K] \omega (| \sigma |, x)$$

Proof. Together with the trajectory $x (t)$ of the equation

$$x (t) = x (t_{i, j-1}) + (t - t_{i, j-1}) f [t_{i, j-1}, \quad x (s: 0 \leqslant s \leqslant t_{i, j-1}), \quad u_{i, j}, v_{i, j}]$$

$$(t \in [t_{i, j-1}, \quad t_{i, j}], \quad u_{i, j} \in P, \quad v_{i, j} \in Q, \quad 1 \leqslant j \leqslant m_i, \quad 1 \leqslant i \leqslant n)$$

we consider the trajectory $y (t)$ defined by the equation

$$y (t) = y (t_{i-1}) + (t - t_{i-1}) \sum_{1 \leqslant j \leqslant m_i} (\Delta_{i, j} / \Delta_i) f [t_{i-1}, \quad y (s: 0 \leqslant s \leqslant t_{i-1}), u_{i, j}, v_{i, j}]$$

$$(t \in [t_{i-1}, t_i], \quad u_{i, j} \in P, \quad v_{i, j} \in Q, \quad 1 \leqslant i \leqslant n)$$

and the initial condition $y (0) = x (0) = x_0$. On the basis of the inequalities

$$\| f [t, x (s: 0 \leqslant s \leqslant t), u, v] - f [\tau, y (s: 0 \leqslant s \leqslant \tau), u, v] \| \leqslant$$

$$\leqslant \| f [t, x (s: 0 \leqslant s \leqslant t, u, v] - f [\tau, x (s: 0 \leqslant s \leqslant \tau), u, v] \| +$$

$$+ \| f [\tau, x (s: 0 \leqslant s \leqslant \tau), u, v] - f [\tau, y (s: 0 \leqslant s \leqslant \tau), u, v] \| \leqslant$$

$$\leqslant \omega (| t - \tau |, x_0) + K \| x (s) - x (s) \|_\tau \quad (3.4)$$

$$\|x(t)-y(t)\| \leqslant 2M|t-\tau| + \|x(\tau)-y(\tau)\|$$

$$\|x(s)-y(s)\|_{t_i} \leqslant 2M|\sigma| + \max_{0\leqslant j\leqslant i}\|x(t_j)-y(t_j)\| \tag{3.5}$$

it is easy to show by induction in i that

$$\max_{0\leqslant j\leqslant i}\|x(t_j)-y(t_j)\| \leqslant b_i \qquad (0\leqslant i\leqslant n) \tag{3.6}$$

$$b_0=0, \quad b_i=(1+K\Delta_i)b_{i-1}+\Delta_i[2KM|\sigma|+\omega(|\sigma|, x_0)]$$

We set

$$c_i(\sigma)=b_i/[2KM|\sigma|+\omega(|\sigma|, x_0)]$$

Then the quantities $c_i(\sigma)$ satisfy the recurrent relations

$$c_0(\sigma)=0, \quad c_i(\sigma)=(1+K\Delta_i)c_{i-1}(\sigma)+\Delta_i \quad (1\leqslant i\leqslant n) \tag{3.7}$$

These relations permit us to represent $c_n(\sigma)$ in the form

$$c_n(\sigma)=(\Delta_1+\ldots+\Delta_n)+K(\Delta_1\Delta_2+\ldots+\Delta_1\Delta_n+$$
$$+\Delta_2\Delta_3+\ldots+\Delta_2\Delta_n+\ldots+\Delta_{n-1}\Delta_n)+\ldots+K^{n-1}\Delta_1\ldots\Delta_n$$

Hence $c_n(\sigma)\leqslant c_n(\sigma(n))$, where $\sigma(n)$ is the covering formed from n intervals $[t_{i-1}, t_i]$ of same length $\Delta_i=\vartheta/n$. Substituting $\Delta_i=\vartheta/n$ into (3.7) we find

$$c_n(\sigma(n))=[1+(1+K\vartheta/n)+\ldots+(1+K\vartheta/n)^{n-1}]\vartheta/n=$$
$$=[(1+K\vartheta/n)^n-1]K\leqslant(e^{K\vartheta}-1)/K$$

This leads to the following estimate:

$$b_n\leqslant[(e^{K\vartheta}-1)/K][2KM|\sigma|+\omega(|\sigma|, x_0]$$

Hence, from (3.5), (3.6) and Condition 6 it follows that

$$\|x(s)-y(s)\|_\vartheta\leqslant\Omega, |F[x(s:0\leqslant s\leqslant\vartheta)]-F[y(s:0\leqslant s\leqslant\vartheta)]\leqslant L\Omega$$

for any

$$(u_{1,1}, v_{1,1}, \ldots, u_{n,m_n}, v_{n,m_n})\in(P\times Q)^m$$

On the basis of Lemma 1 we obtain

$$|V_{\sigma'}^{+}(x_0)-V_{\sigma,\sigma'}^{\pm}(x_0)|=|\mathrm{VAL}_m^{\pm}F[x(s: 0\leqslant s\leqslant\vartheta)]-\mathrm{VAL}_m^{\pm}F[y(s: 0\leqslant s\leqslant\vartheta)]|\leqslant L\Omega$$

The lemma is proved.

We set

$$\Sigma_*(\sigma)=\{\sigma'': \sigma''\in\Sigma, l(\sigma'')=l(\sigma)\}$$

We denote the general element of the set $\Sigma_*(\sigma)$ by σ_*. Let $[\tau_{i-1}, \tau_i]$ be sequentially contiguous intervals comprising the covering σ_*

$$\delta_i=\tau_i-\tau_{i-1}, \qquad \rho[\sigma, \sigma_*]=\max_{0\leqslant i\leqslant n}|t_i-\tau_i|$$

Lemma 4. Let Conditions 3 – 6 be fulfilled. Then there exists a non-negative scalar function $D(\sigma, \sigma_*, x)$ such that for any $\sigma\in\Sigma$

$$|V_\sigma^{\pm}(x)-V_{\sigma*}^{\pm}(x)|\leqslant LD(\sigma, \sigma_*, x) \tag{3.8}$$

$$\lim_{\rho[\sigma,\sigma_*]\to 0} D(\sigma, \sigma_*, x)\leqslant 2Me^{K\vartheta}|\sigma| \tag{3.9}$$

Proof. Together with the trajectory $x(t)$ of Eq. (2.1) we consider the trajectory $y(t)$ defined by the equation

$$y(t) = y(\tau_{i-1}) + (t - \tau_{i-1}) f[\tau_{i-1}, y(s: 0 \leqslant s \leqslant \tau_{i-1}), u_i, v_i]$$
$$(t \in [\tau_{i-1}, \tau_i], \quad u_i \in P, \quad v_i \in Q, \quad 1 \leqslant i \leqslant n)$$

and the initial condition $y(0) = x(0) = x_0$. Using (3.4) jointly with the inequalities

$$|\Delta_i - \delta_i| \leqslant 2\rho[\sigma, \sigma_*]$$
$$\|x(s) - y(s)\|_{t_i} \leqslant 2M|\sigma| + M\rho[\sigma, \sigma_*] + \max_{0 \leqslant j \leqslant i} \|x(t_j) - y(\tau_j)\| \tag{3.10}$$

by an induction in i it is not difficult to establish that

$$\max_{0 \leqslant j \leqslant i} \|x(t_j) - y(\tau_j)\| \leqslant d_i \qquad (0 \leqslant i \leqslant n) \tag{3.11}$$

$$d_0 = 0; \quad d_i = (1 + K\Delta_i) d_{i-1} + \Delta_i \omega(\rho[\sigma, \sigma_*], x_0) + $$
$$+ \Delta_i KM(2|\sigma| + \rho[\sigma, \sigma_*]) + 2M\rho[\sigma, \sigma_*] \tag{3.12}$$

We set

$$D(\sigma, \sigma_*, x_0) = d_n(\sigma, \sigma_*, x_0) + M(2|\sigma| + \rho[\sigma, \sigma_*])$$

Then by virtue of (3.10) and (3.11)

$$\|x(s) - y(s)\|_\vartheta \leqslant D(\sigma, \sigma_*, x_0)$$
$$|F[x(s: 0 \leqslant s \leqslant \vartheta)] - F[y(s: 0 \leqslant s \leqslant \vartheta)]| \leqslant LD(\sigma, \sigma_*, x_0)$$

for all $(u_1, v_1, \ldots, u_n, v_n) \in (P \times Q)^n$. From this and from Lemma 1 we derive (3.8).

To be convinced of the validity of (3.9) we consider the quantities $d_i^* = d_i^*(\sigma)$ defined by the formulas

$$d_0^* = 0, \quad d_i^* = (1 + K\Delta_i) d_{i-1}^* + 2\Delta_i KM|\sigma| \quad 1 \leqslant i \leqslant n \tag{3.13}$$

Comparing (3.12) and (3.13) and keeping in mind that

$$\omega(\lambda, x_0) \to 0 \quad \text{as} \quad \lambda \to 0 \tag{3.14}$$

we conclude that

$$d_n(\sigma, \sigma_*, x_0) \to d_n^*(\sigma) \quad \text{for} \quad \rho[\sigma, \sigma_*] \to 0$$

Now (3.9) follows from the estimate

$$d_n^*(\sigma) \leqslant (e^{K\vartheta} - 1) 2M|\sigma|$$

L e m m a 5. Let Conditions 2 – 6 be fulfilled. Let $\sigma' \supseteq \sigma$. Then

$$V_\sigma^+(x) \geqslant V_{\sigma, \sigma'}^+(x), \qquad V_\sigma^-(x) \leqslant V_{\sigma, \sigma'}^-(x)$$

P r o o f. The inequalities

$$V_{\sigma, i}^+ \geqslant V_{\sigma, \sigma', i}^+, \qquad V_{\sigma, i}^- \leqslant V_{\sigma, \sigma', i}^- \tag{3.15}$$

are true for $i = n$. Their validity in the general case is established by the same inductive reasoning which Fleming employed to prove Lemma 3 in [2]. When $i = 0$ the inequalities (3.15) turn into the relations called for in the lemma.

T h e o r e m 2. Let Conditions 2 – 6 be fulfilled. Then the limits (3.1) exist and the convergence is uniform in x on every bounded subset of space X and the limit functions $V_{(x)}^{\pm}$ satisfy a Lipschitz condition on X with the constant $Le^{K\vartheta}$.

P r o o f. Let

$$\Sigma^*(\sigma) = \{\sigma'': \sigma'' \in \Sigma, \ l(\sigma'') \geqslant l(\sigma)\}, \quad \sigma^* \in \Sigma^*(\sigma)$$
$$\Sigma(\sigma, \sigma^*) = \{\sigma'': \sigma'' \in \Sigma, \ \sigma'' \subseteq \sigma^*, \quad l(\sigma'') = l(\sigma)\}$$

Let $\sigma_*(\sigma, \sigma^*)$ be an arbitrary covering from $\Sigma(\sigma, \sigma^*)$, subject to the relations

$$\sigma_*(\sigma, \sigma^*) \subseteq \sigma^* \tag{3.16}$$
$$l(\sigma_*(\sigma, \sigma^*)) = l(\sigma) \tag{3.17}$$

$$\rho[\sigma, \sigma_*(\sigma, \sigma^*)] = \min_{\sigma'' \in \Sigma(\sigma, \sigma^*)} \rho[\sigma, \sigma''] \tag{3.18}$$

On the basis of Lemmas 3 – 5, (3.16) and (3.17),

$$V_{\sigma*}^+(x) \leqslant V_\sigma^+(x) + L\Omega + LD(\sigma, \sigma_*(\sigma, \sigma^*), x)$$

In the following three relations the limit in the left-hand side ranges over all $\sigma^* \in \Sigma^*(\sigma), |\sigma^*| \to 0$. According to (3.16) - (3.18),

$$\lim \rho[\sigma, \sigma_*(\sigma, \sigma^*)] = 0$$

Therefore, by virtue of (3. 9),

$$\overline{\lim}\, V_{\sigma*}^+(x) \leqslant V_\sigma^+(x) + L\Omega + 2LMe^{K\vartheta}|\sigma|$$

But for any fixed $\sigma \in \Sigma$

$$\overline{\lim}\, V_{\sigma*}^+ = \overline{\lim}_{|\sigma| \to 0} V_\sigma^+(x)$$

Consequently, for every $\sigma \in \Sigma$

$$\overline{\lim}_{|\sigma| \to 0} V_\sigma^+(x) \leqslant V_\sigma^+(x) + L\Omega + 2LMe^{K\vartheta}|\sigma|$$

However, in view of (3.14) the sum $\Omega + 2\, Me^{K\vartheta}|\sigma|$ tends to zero as $|\sigma| \to 0$. Hence

$$\overline{\lim_{|\sigma| \to 0}}\, V_\sigma^+(x) \leqslant \underline{\lim}_{|\sigma| \to 0} V_\sigma^+(x)$$

By the same token the existence of the limit $V^+(x)$ is established. The proof of the existence of the limit $V^-(x)$ is carried out analogously. The rest of the theorem follows from Lemma 2.

4. Here we show that when Conditions 1 – 6 are fulfilled there exists a generalized value $V = V^+ = V^-$ of game Γ.

Lemma 6. Let conditions 1 – 6 be fulfilled. Then for each $\sigma \in \Sigma$ the limits

$$V_{\sigma,0,i}[y(s: 0 \leqslant s \leqslant t_i)] = \lim_{\sigma' \supseteq \sigma, \|\sigma'\| \to 0} V_{\sigma,\sigma',i}^{\pm}[y(s: 0 \leqslant s \leqslant t_i)]$$

exist and are equal, the convergence is uniform in $y(s)$ on every compact subset of the space $C[0, t_i]$, and the limit functions $V_{\sigma,i,0}$ satisfy a Lipschitz condition on $C[0, t_i]$ with a constant $Le^{K\vartheta}$.

Proof. The lemma is true for $i = n$. Reasoning inductively, we assume that the lemma is valid for some i. Under this assumption we prove the lemma for $i - 1$. First of all we note that on the basis of Lemma 2 it is sufficient to convince ourselves that for every $y(s) \in C[0, t_{i-1}]$ the limits $V_{\sigma,0,i-1}$ exist and are equal. Having fixed $y(s) \in C[0, t_{i-1}]$, by B we denote the collection of functions $z(s) \in C[0, t_i]$, each of which coincides with $y(s)$ on $[0, t_{i-1}]$ while on $[t_{i-1}, t_i]$ are subject to a Lipschitz condition in s with constant M. We set

$$\varepsilon^{\pm} = \max_{z(s) \in B} |V_{\sigma,0,i}[z(s: 0 \leqslant s \leqslant t_i)] - V_{\sigma,\sigma',i}^{\pm}[z(s: 0 \leqslant s \leqslant t_i)]|$$

Then, according to Lemma 1, the deviation

$$|V_{\sigma,\sigma',i-1}^{\pm}[y(s: 0 \leqslant s \leqslant t_{i-1})] - \mathrm{VAL}_{m_i}^{\pm} V_{\sigma,0,i}[y^*(\cdot)]| =$$
$$= |\mathrm{VAL}_{m_i}^{\pm} V_{\sigma,\sigma',i}^{\pm}[y^*(\cdot)] - \mathrm{VAL}_{m_i}^{\pm} V_{\sigma,0,i}[y^*(\cdot)]|$$

does not exceed $\varepsilon^{\pm}$

Let us now consider, on the interval $[t_{i-1}, t_i]$ a certain auxiliary differential game Γ^*.

The game Γ^* is described by the equation $x^{\cdot}(t) = f^*(u, v) \equiv f[t_{i-1}, y(s: 0 \leqslant s \leqslant t_{i-1}), u, v]$, $u \in P$, $v \in Q$ with initial condition $x(t_{i-1}) = y(t_{i-1})$. The payoff in Γ^* is the functional

$$F^*[y(t_{i-1}),\ x(t_i)] = V_{\sigma, 0, i}[\xi(s: 0 \leqslant s \leqslant t_i)]$$

where $\xi(t)$ is a prolongation of the function $y(t) \in C[0, t_{i-1}]$ onto the interval $[0, t_i]$, given on $[t_{i-1}, t_i]$ by the formula

$$\xi(t) = y(t_{i-1}) + [(t - t_{i-1}) / \Delta_i][x(t_i) - y(t_{i-1})].$$

Since Γ^* satisfies Conditions 1 – 6 and does not contain aftereffect elements, to it we can apply the results of [8, 9]. On the basis of these results it is easy to establish that the limits

$$\lim \mathrm{VAL}^{\pm}_{m_i} F^*[y(t_{i-1}), x(t_i)] \equiv \lim \mathrm{VAL}^{\pm}_{m_i} V_{\sigma, 0, i}[x^*(s: 0 \leqslant s \leqslant t_i)] \equiv$$
$$\equiv \lim \mathrm{VAL}^{+}_{m_i} V_{\sigma, 0, i}[y^*(s: 0 \leqslant s \leqslant t_i)]$$

exist and are equal (the limits are taken over all $\sigma' \supseteq \sigma$, $|\sigma'| \to 0$). Since by the inductive assumption $\varepsilon^{\pm} \to 0$ as $|\sigma'| \to 0$, the limits $V_{\sigma, 0, i-1}$ also exist and are equal. The lemma is proved.

Theorem 3. Let Conditions 1 – 6 be fulfilled. Then $V^+(x) = V^-(x)$.

Proof. According to Lemma 3, for any $\sigma' \supseteq \sigma$

$$|V^+_{\sigma'}(x) - V^-_{\sigma'}(x)| \leqslant |V^+_{\sigma, \sigma'}(x) - V^-_{\sigma, \sigma'}(x)| + 2L\Omega$$

In correspondence with Lemma 6, considered for $i = 0$, the deviation $|V^+_{\sigma, \sigma'}(x) - V^-_{\sigma, \sigma'}(x)|$ tends to zero as $|\sigma'| \to 0$. Therefore, by Theorem 2 the deviation $|V^+(x) - V^-(x)|$ does not exceed $2L\Omega$. But $\Omega \to 0$ when $|\sigma| \to 0$. Consequently, $V^+(x) = V^-(x)$.

5. Let M (N) be the collection of measures μ (ν) given on a σ-algebra of Borel subsets of set P (Q) and normed on this set,

$$\mu(P) = \int d\mu = 1 \left(\nu(Q) = \int d\nu = 1\right)$$

We set

$$f[p, \mu, \nu] = \iint f[p, u, v]\, d\mu d\nu$$

We denote the original differential game, considered in the mixed formulation [8, 9] by G. In the mixed formulation the development of the game is described by the equation

$$x^{\cdot}(t) = f[t, x(s: 0 \leqslant s \leqslant t), \mu, \nu], \quad \mu \in \mathrm{M}, \ \nu \in \mathrm{N}$$

Thus, in G the measures $\mu \in \mathrm{M}$, $\nu \in \mathrm{N}$ are actually the players' controls. In this connection we remark that the sets M, N are weakly compact (see [14], p. 791) and convex. If in the definitions of Sect. 2 we carry out the replacement $u, v, P, Q \to \mu, \nu, \mathrm{M}, \mathrm{N}$, we arrive at the corresponding definitions for differential game G. We denote by $G^{\pm}_{\sigma}$ the majorant and the minorant of the mixed multistage games corresponding to the covering $\sigma \in \Sigma$. The values of these games we denote by $U^{\pm}_{\sigma}$.

Theorem 4. Let Conditions 3 – 6 be fulfilled. Then in the games $G^{\pm}_{\sigma}$ there exists a saddle point, the values $U^{\pm}_{\sigma}$ of these games satisfy the relation

$$U^{\pm}_{\sigma}(x_0) = \mathrm{VAL}^{\pm}_{n} F[x(s: 0 \leqslant s \leqslant \theta)] \qquad (n = l(\sigma))$$

the limits

$$U^{\pm}(x) = \lim_{|\sigma| \to 0} U^{\pm}_{\sigma}(x)$$

exist and are equal, the convergence is uniform in x on every bounded subset of space X, and the functions $U^{\pm}_{\sigma}$ and their limit values $U^+(x) = U^-(x)$ are subject to a

Lipschitz condition on X with a constant $Ie^{K\vartheta}$.

6. Let conditions 2 – 6 be fulfilled. Then there simultaneously exist the generalized values $V^{\pm}$ of the differential games $\Gamma^{\pm}$ and the generalized value $U = U^{+} = U^{-}$ of differential game G. It is easy to verify that these values are connected by the inequality $V^{+} \geqslant U \geqslant V^{-}$ Consequently, when Conditions 1 – 6 are fulfilled, $V^{+} = U = V^{-} = V$.

7. We consider the important particular case when the equation of motion (1.1) does not contain aftereffect elements, i.e., has the form

$$x^{\cdot}(t) = f[t,\ x(t),\ u,\ v] \qquad (u \in P,\ v \in Q) \tag{7.1}$$

In this case it is of interest to elicit these additional conditions on the payoff functional under whose fulfillment the differential game with complete information on the current position $(t,\ x(t))$ has a (generalized) value in pure or mixed strategies.

A functional F is called quasi-additive if there exists a scalar function $\Phi(\alpha,\ \beta)$, continuous on $(-\infty,\ \infty) \times (-\infty,\ \infty)$ possessing the following properties:

$$\Phi(\alpha,\ \beta) \leqslant \Phi(\alpha,\ \beta') \quad \text{for all} \quad \alpha,\ \beta \leqslant \beta$$

$$F[x(s: a \leqslant s \leqslant b)] =$$

$$= \Phi\,[(F\,(x\,(s: a \leqslant s \leqslant \tau)],\ F\,[x\,(s: \tau \leqslant s \leqslant b)])$$

for all $0 \leqslant a \leqslant \tau \leqslant b \leqslant \vartheta,\ x(s) \in C\,[0,\ \vartheta]$ Let $h(x)$ be a scalar function, continuous on X. Then the functionals

$$F = h(x(\vartheta)), \quad F = \min_{0 \leqslant s \leqslant \vartheta} h(x(s)), \quad F = \int_0^{\vartheta} h(x(s))\,ds$$

are quasi-additive: in the first case $\Phi(\alpha.\ \beta) = \beta$, in the second case $\Phi(\alpha,\ \beta) = \min\{\alpha,\ \beta\}$ and in the third case $\Phi(\alpha,\ \beta) = \alpha + \beta$.

Theorem 5. Let Conditions 3 – 6 be fulfilled. We assume that the functional F is quasi-additive and that the equation of motion has the form (7.1). Then in the games $\Gamma_{\sigma}^{\pm}$ $(G_{\sigma}^{\pm})$ there exists a saddle point formed by the strategies

$$\{u_1^{\circ}, \ldots, u_n^{\circ}\}, \qquad \{v_1^{\circ}, \ldots, v_n^{\circ}\}$$

$$(\{\mu_1^{\circ}, \ldots, \mu_n^{\circ}\}, \qquad \{\nu_1^{\circ}, \ldots, \nu_n^{\circ}\})$$

each component of which is independent of $x\,(s: 0 \leqslant s < t_{-1})$ for $i > 1$

The proof is obtained from the fact that under the theorem's hypotheses the functions $V_{\sigma,i}^{\pm}$ can be determined by the formulas

$$V_{\sigma,n}^{\pm}[y(s: 0 \leqslant s \leqslant \vartheta)] = F\,[y(s: 0 \leqslant s \leqslant \vartheta)]$$

$$V_{\sigma,i-1}^{\pm}[y(s: 0 \leqslant s \leqslant t_{i-1})] = \mathrm{VAL}^{\pm}\Phi\,(F\,[y\,(s: 0 \leqslant s \leqslant t_{i-1})],\ V_{\sigma,i}^{\pm}[y_*(s: t_{i-1} \leqslant s \leqslant t_i)]) =$$

$$= \Phi\,(F\,[y\,(s: 0 \leqslant s \leqslant t_{i-1})],\ \mathrm{VAL}^{\pm}V_{\sigma,i}^{\pm}[y_*(s: t_{i-1} \leqslant s \leqslant t_i)]) \tag{7.2}$$

This possibility, in its own turn, is explained by the equalities

$$V_{\sigma,0}^{\pm}[x_0] = \mathrm{VAL}_n^{\pm}F\,[x\,(s: 0 \leqslant s \leqslant \vartheta)] \quad (n = l\,(\sigma))$$

which are easily derived from (7.2) and from the expansion

$$F\,[x\,(s: 0 \leqslant s \leqslant \vartheta)] = \Phi\,(F\,[x\,(s: t_0 \leqslant s \leqslant t_1)]\ \Phi\,(\cdots\Phi\,(F\,[x\,(s: t_{n-2} \leqslant s \leqslant t_{n-1})]$$

$$F\,[x\,(s: t_{n-1} \leqslant s \leqslant t_n)])\cdots))$$

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